

# Escape and trapping of low-frequency gravitationally lensed rays by compact objects within plasma

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## ABSTRACT

We consider the gravitational lensing of rays emitted by a compact object (CO) within a distribution of plasma with power-law density  $\propto r^{-h}$ . For the simplest case of a cloud of spherically symmetric cold non-magnetized plasma, the diverging effect of the plasma and the converging effect of gravitational lensing compete with one another. When  $h < 2$ , the plasma effect dominates over the vacuum Schwarzschild curvature, potentially shifting the radius of the unstable circular photon orbit outside the surface of the CO. When this occurs, we define two relatively narrow radio-frequency bands in which plasma effects are particularly significant. Rays in the escape window have  $\omega_0 < \omega \leq \omega_+$  and are free to propagate to infinity from the CO surface. To a distant observer, the visible portion of the CO surface appears to shrink as the observed frequency is reduced, and vanishes entirely at  $\omega_0$ , in excess of the plasma frequency at the CO surface. We define the anomalous propagation window for frequencies  $\omega_- < \omega \leq \omega_0$ . Rays emitted from the CO surface within this frequency range are dominated by optical effects from the plasma and curve back to the surface of the CO, effectively cloaking the star from distant observers. We conclude with a study of neutron star (NS) compactness ratios for a variety of nuclear matter equations of state (EoS). For  $h = 1$ , NSs generated from stiff EoS should display significant frequency dependence in the EW, and lower values of  $h$  with softer EoS can also show these effects.

**Key words:** gravitation – plasmas – stars: neutron – pulsars: general

## 1 INTRODUCTION

Gravitational lensing and plasma effects were first considered in the literature by [Synge \(1960\)](#), and later applied by [Muhleman & Johnston \(1966\)](#) and [Muhleman, Ekers & Fomalont \(1970\)](#), who calculated the effect of the solar plasma on the deviation of background starlight. The combination of gravitational lensing and plasma effects were significantly expanded on by [Perlick \(2000\)](#) and [Bisnovatyi-Kogan & Tsupko \(2009\)](#), leading to studies that range from the scale of stellar mass black holes in X-ray binaries ([Bisnovatyi-Kogan & Tsupko 2010](#)) to galaxy-scale lenses ([Er & Mao 2014](#)). This work has been extended to include rotation using the Kerr metric ([Morozova, Ahmedov & Tursunov 2013](#)). The effect of plasma on the shadows of black holes in general relativity has also been explored ([Atamurotov, Ahmedov & Abdujabbarov 2015](#); [Perlick, Tsupko & Bisnovatyi-Kogan 2015](#); [Abdujabbarov et al. 2016a](#)), as well as near exotic ob-

jects ([Abdujabbarov et al. 2015, 2016b](#)) and in  $f(R)$  gravity ([Dastan, Saffari & Soroushfar 2016](#)). For a detailed overview of gravitational lensing effects within distributions of plasma, we refer to the thorough review provided by [Bisnovatyi-Kogan & Tsupko \(2015\)](#).

The plasma frequency  $\omega_e$  defines a low-frequency cutoff for electromagnetic wave propagation. The index of refraction  $n(\omega, r)$  for radiation below this frequency is imaginary, which indicates that such low-frequency radiation is absorbed by the plasma. Generally the effect of plasma on ray trajectories near the compact object (CO) is expressed as an additional term in the effective potential. [Kulsrud & Loeb \(1992\)](#) showed that a constant plasma density provides an effective mass for light rays through this potential term. However, inhomogeneous distributions of plasma can generate more interesting effects ([Tsupko & Bisnovatyi-Kogan 2013](#)). [Rogers \(2015\)](#) considered rays passing a massive object sheathed in a distribution of cold, non-magnetized plasma with a power-law density distribution  $r^{-h}$  for integer  $0 < h < 3$ , and presented detailed calculations of the effect of the plasma with  $h = 3$  on the pulse profiles of a central neutron star (NS) with two emitting polar caps. Due

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to the dipolar magnetic field of the NS, the  $h = 3$  case was primarily investigated.

In this work, we investigate the consequences of low-frequency emission from a CO sheathed in plasma with  $h < 2$ . For this case, the plasma term in the effective potential drops off more slowly than the Schwarzschild vacuum term. When this occurs the diverging-lens behaviour of the plasma dominates for low-frequency rays, such that a potential maximum exists external to the stellar radius  $R$  for sufficiently compact stars. Thus, low-frequency radiation is affected by turning points of the effective potential, causing rays which would usually escape the surface of the CO in the vacuum case to turn back towards the stellar surface. A wide range of other novel behaviour exists for rays that escape the CO surface with frequencies  $\omega_0 < \omega \leq \omega_+$ . We define this frequency range as the escape window (EW), and the trapped rays with  $\omega_- < \omega \leq \omega_0$  are within the anomalous propagation window (APW). Ray trapping in the APW is analogous to the anomalous propagation of radio signals in the Earth's atmosphere, and has profound effects on the appearance of a CO to distant observers, such as reducing the apparent radius of a CO as a function of frequency. In fact, the CO appears to vanish entirely as the observed frequency approaches  $\omega_0$ , effectively 'cloaking' the stellar surface from a distant observer when observed at low frequencies. These frequency windows do not exist when the plasma density drops off more rapidly than the vacuum term in the effective gravitational potential for null geodesics in the Schwarzschild space-time.

We review the details of the geometrical optics approach to gravitational lensing within a plasma distribution in Section 2, provide the details of the frequency windows in Section 2.1 and derive the maximum impact parameter as a function of frequency in Section 2.2. We discuss our results in Section 3 and summarize our conclusions in Section 4.

## 2 THEORY

Previous work (Rogers 2015) examined modifications to ray trajectories introduced by a simple choice of dispersion relation in the case of spherical symmetry, following the general approach and notation of Tsupko & Bisnovatyi-Kogan (2013). We will briefly review the assumptions behind this physical scenario here.

We make use of the Schwarzschild metric to describe the space-time surrounding a spherically symmetric CO,

$$ds^2 = -A(r)dt^2 + \frac{dr^2}{A(r)} + r^2(d\theta^2 + \sin^2\theta d\phi^2) \quad (1)$$

with the metric function

$$A(r) = 1 - \frac{r_g}{r} \quad (2)$$

and abbreviating  $r_g = 2M$ . In these definitions and for the rest of this paper, we use units which have  $G = c = \hbar = 1$ . Let us assume the index of refraction for a cold plasma

$$n^2(r, \omega) = 1 - \frac{\omega_e^2}{\omega^2}, \quad (3)$$

with plasma frequency

$$\omega_e^2(r) = \frac{4\pi e^2 N(r)}{m} \quad (4)$$

that has a plasma particle charge and mass given by  $e$  and  $m$ , respectively, as well as the plasma number density  $N(r)$ . An often used plasma density in the literature is a radial power-law (Tsupko & Bisnovatyi-Kogan 2013), such that

$$N(r) = \frac{N_0}{r^h}. \quad (5)$$

This puts the plasma frequency into a simple form,

$$\omega_e^2(r) = \frac{k}{r^h}, \quad (6)$$

where  $k$  is a constant and  $h$  is the power-law exponent. For simplicity, we set the constant  $k = 1$  in our numerical calculations. Throughout the text we refer to frequencies detected by a distant observer with the subscript  $\infty$ , from the redshift relation

$$\omega(r) = \frac{\omega_\infty}{A(r)^{1/2}} \quad (7)$$

and do not explicitly state the radial dependence of  $\omega$ . The Hamiltonian for rays in curved space-time and under the effects of an optical medium with index of refraction (Synge 1960) using cold non-magnetized plasma is

$$H(x^i, p_i) = \frac{1}{2} (g^{ij} p_i p_j + \omega_e^2) = 0. \quad (8)$$

The equations of motion are given in terms of an arbitrary curve parameter  $\lambda$ :

$$\frac{dx^i}{d\lambda} = \frac{\partial H}{\partial p_i} = g^{ij} p_j \quad (9)$$

$$\frac{dp_i}{d\lambda} = -\frac{\partial H}{\partial x^i} = -\frac{1}{2} g^{jk}_{,i} p_j p_k - \frac{1}{2} (\omega_e^2)_{,i}. \quad (10)$$

From these relationships equation 10 immediately gives

$$\frac{dp_t}{d\lambda} = \frac{dp_\phi}{d\lambda} = \frac{dp_\theta}{d\lambda} = 0. \quad (11)$$

The vanishing derivatives show that the  $t$ ,  $\theta$  and  $\phi$  components must be constant. By spherical symmetry, we choose to work in the equatorial plane so take  $\theta = \pi/2$ . The  $t$  and  $\phi$  quantities are interpreted as the energy and angular momentum of a ray,

$$p_t = -E = -\omega_\infty \quad (12)$$

and

$$p_\phi = L = \omega_\infty b, \quad (13)$$

where  $b$  is the impact parameter of an escaping ray. Finally, the derivative of the radial momentum is

$$\frac{dp_r}{d\lambda} = -\frac{M}{r^2 A^2(r)} E^2 - \frac{M}{r^2} p_r^2 + \frac{L^2}{r^3} + \frac{1}{2} \frac{kh}{r^{h+1}}. \quad (14)$$

We can find an expression for the radial momentum  $p_r$  directly from the vanishing of the Hamiltonian, such that

$$p_r = \pm \frac{L}{A(r)} \left[ \frac{E^2}{L^2} - A(r) \left( \frac{1}{r^2} + \frac{1}{L^2} \frac{k}{r^h} \right) \right]^{1/2} \quad (15)$$

and the sign of  $p_r$  depends on the ray trajectory, with positive indicating an outgoing ray and negative for an incoming ray. The coordinate derivatives using equation 9 give

$$\frac{dt}{d\lambda} = \frac{E}{A(r)} \quad (16)$$

$$\frac{dr}{d\lambda} = A(r)p_r \quad (17)$$

$$\frac{d\phi}{d\lambda} = \frac{L}{r^2} \quad (18)$$

and working in the equatorial plane implies

$$\frac{d\theta}{d\lambda} = 0. \quad (19)$$

For an initial position  $x_{\text{initial}} = (t, r, \theta, \phi)$ , ray energy  $E$  and angular momentum  $L$ , as well as a choice of sign for  $p_r$ , equations 11 to 19 form a system of equations that are easily solved using an integration scheme for second-order ordinary differential equations, such as the fourth-order Runge–Kutta method. The solution to this system gives points along the path of a ray trajectory launched from  $x_{\text{initial}}$  in the direction of our choice. We stop integration when the ray returns to the surface of the CO ( $r < R$ ), or when it escapes to a sufficiently great distance ( $r \geq 100R$ ).

The trajectory of a ray near the CO is described in terms of an effective potential. We denote  $\dot{r} = dr/d\lambda$ , and use equations 15 and 17 to give an energy conservation equation

$$\dot{r}^2 = E^2 - V_{\text{eff}}(r) \quad (20)$$

with

$$V_{\text{eff}} = \left(1 - \frac{2M}{r}\right) \left[\frac{L^2}{r^2} + \frac{k}{r^h}\right], \quad (21)$$

where the vacuum contribution is the first term in the square brackets and the second term is entirely due to the plasma. Setting the square of the ray energy equal to the effective potential gives the propagation condition

$$\omega_\infty \geq \sqrt{V_{\text{eff}}(r)} \quad (22)$$

which must be satisfied at every point along the trajectory for rays to escape from the surface of the CO and travel to infinity. The impact parameter of an escaping ray with asymptotic frequency  $\omega_\infty$  is

$$b(r) = \frac{rn(r)}{A(r)^{1/2}} \sin(\delta). \quad (23)$$

This ray will intersect the radial normal vector  $\hat{r}$  at  $r$  with an angle  $\delta$ . A ray passing  $r$  with a grazing incidence  $\delta = \pi/2$  and escaping to reach an observer at infinity defines the maximum impact parameter,  $b_{\text{max}}$  at a particular frequency. All rays that pass  $r$  with impact parameter  $b < b_{\text{max}}$  will intersect  $\hat{r}$  at smaller angles.

In analogy to equation 7, we use the effective redshift formula to define the plasma frequency at infinity,

$$\omega_{\infty e} = A(r_{\text{max}})^{1/2} \omega_e(r_{\text{max}}), \quad (24)$$

where  $r_{\text{max}}$  is the position of the maximum density along the path of a given ray. The rays that propagate through plasma that are visible to distant observers require  $\omega_\infty > \omega_{\infty e}$  everywhere along their trajectory. Note that the plasma frequency at infinity is equivalent to the square root of the effective potential (equation 21) with  $L = 0$ . This means that radially directed rays with sufficiently low asymptotic frequency will be absorbed since the index of refraction  $n(\omega, r)$  vanishes for rays at the asymptotic plasma frequency. These rays have

absorption points rather than turning points. However, rays with finite  $L$  at these same frequencies can propagate freely since they remain above the plasma frequency cutoff at all times due to the contribution of  $L$  to the effective potential.

## 2.1 Frequency windows

For plasma density distributions with power-law index  $h < 2$ , the propagation of low-frequency rays is complicated by the presence of a turning point that can be external to the stellar surface. The dependence between the radius of the photon sphere and  $h$  was discussed by Perlick, Tsupko & Bisnovatyi-Kogan (2015) in their work on black hole shadows in plasma. However, the effect has additional implications for objects with surfaces that produce low-frequency emission.

We define the EW as a range of frequencies  $\omega_0 < \omega \leq \omega_+$  within which rays are strongly affected by the plasma, but are still free to escape to an observer at infinity. At  $\omega_+$ , the effective potential allows a circular orbit at the exact surface of the CO for the maximum impact parameter given by equation 23 with  $r = R$ . This condition remains valid for higher frequencies  $\omega_+ < \omega$  as these have circular orbit radii within the stellar surface. At the lower limit of the EW only the fiducial ray with frequency  $\omega_0$  escapes to a distant observer.

To describe these effects quantitatively, let the circular orbit radius be called  $r_c$ , the position of the potential maximum where the derivative of the effective potential vanishes. This condition gives the critical angular momentum required for a circular orbit in terms of  $h$ ,

$$L_c^2(r_c) = \frac{Mr_c^2}{(r_c - 3M)} \frac{k}{r_c^{h-1}} \left( \frac{h+1}{r_c} - \frac{h}{2M} \right), \quad (25)$$

with  $L_c = \omega_\infty b_c$  in terms of the observed frequency at infinity, and we write the critical impact parameter as  $b_c = b(r_c)$ . The upper frequency limit of the EW is found when the circular orbit and stellar surface coincide at  $r_c = R$ , with the corresponding angular momentum  $L_+ = L_c(r_c) = L_c(R)$ , and  $b_c = b(r_c) = b(R)$ . Above this limit,  $L > L_+$ , the circular orbit is within the stellar surface and all rays are generally free to escape to infinity with maximum impact parameter  $b(R)$ . For this reason, we will restrict ourselves to cases for which  $L \leq L_+$ , in which ray paths are significantly altered from their vacuum behaviour. The angular momentum  $L_+$  can be used to define a corresponding asymptotic frequency which we call the upper limit of the EW,  $\omega_{\infty+}$ . With the effective potential (equation 21), and using the equality in the propagation condition (equation 22), we find an expression for the corresponding asymptotic frequency

$$\omega_{\infty+} = \left(1 - \frac{2M}{R}\right) \left[ \left(1 - \frac{h}{2}\right) \frac{k}{R^{h-1}} \frac{1}{(R - 3M)} \right]^{1/2}. \quad (26)$$

From this expression, we see that the upper frequency limit vanishes when  $h = 2$ , and real values of the upper frequency limit exist only for plasma distributions with  $h < 2$ .

Rays with frequencies  $\omega < \omega_+$  can enter a circular orbit outside the CO surface. The presence of the unstable circular orbit external to the stellar surface implies a maximum in the potential. Therefore a ray escaping the CO surface must have energy in excess of this maximum to escape and avoid a turning (absorption) point in the trajectory. Thus, the

effective potential maximum defines the lower limit of the EW,  $\omega_0$ , which a distant observer detects as an asymptotic frequency

$$\omega_{\infty 0} = \sqrt{V_{\text{eff}}(r_c)}. \quad (27)$$

This is the threshold frequency for rays to escape the stellar surface.

The existence of the EW provides a constraint on the maximum compactness ratio of a CO,  $R/r_g$ . Let us consider a radially directed ray ( $\delta = 0$ ) that is at the threshold of escape. Equation 25 vanishes for a ray that is radially directed. With this condition the position of the potential maximum for general  $h$  is

$$r_c = r_g \left( \frac{h+1}{h} \right). \quad (28)$$

The existence of a turning point at or external to the CO surface  $r_c \geq R$  is necessary for the existence of the EW. Let us set  $r_c = R$  for the most conservative limiting case. Combined with the  $h < 2$  condition implied by equation 26, we find a minimum compactness ratio

$$\frac{R}{r_g} \geq \frac{3}{2}. \quad (29)$$

This compactness ratio is a realistic lower bound for NSs, which we will discuss further in Section 3.

Finally, we also note the simplification introduced for radially directed rays at the minimum EW frequency  $\omega_0$ . Using equation 28 in equation 27 gives the lower EW limit for a radially directed ray,

$$\omega_{\infty 0} = \left[ \frac{k}{h+1} \left( \frac{h}{2M(h+1)} \right)^h \right]^{1/2}. \quad (30)$$

In the specific case  $h = 1$  and  $r_c = 4M$ ,

$$\omega_{\infty 0} = \sqrt{\frac{k}{8M}}. \quad (31)$$

This is the minimum frequency limit for rays to escape along a radial trajectory, and equals the asymptotic plasma frequency (equation 24).

For rays with frequencies lower than the EW, we define the APW, in which the diverging effect of the plasma dominates ray propagation. The APW is the range of frequencies given by  $\omega_- < \omega \leq \omega_0$ . Since  $r_c > R$  implies a potential maximum of  $V_{\text{eff}}(r_c) = \omega_{\infty 0}^2$ , the APW is bounded from below by

$$\omega_{\infty -} = \sqrt{V_{\text{eff}}(R)}. \quad (32)$$

Rays in the APW that are emitted by the CO have frequencies larger than the local plasma frequency for radii  $R < r$ , but below the asymptotic plasma frequency at  $r_c$  required for escape. Thus, these rays are effectively trapped in the environment of the CO. Rays in the APW reach a maximum distance  $r_- < r_c$  where they encounter a turning point due to the potential barrier and subsequently reverse direction to curve back to the CO surface. Rays external to the circular orbit radius with frequencies in the APW that approach the CO from infinity reach a minimum turning distance  $r_+$  and are reflected away from the CO by the potential boundary, returning to infinity. To find the external turning radius as a

function of impact parameter, let us consider the polynomial expression for the turning points (Rogers 2015),

$$r^{h+1} - b^2 r^{h-1} + 2Mb^2 r^{h-2} - \frac{k}{\omega_{\infty}^2} r + 2M \frac{k}{\omega_{\infty}^2} = 0. \quad (33)$$

For a given  $b$ , the solutions of equation 33 give the external turning points  $r_+$ . For rays emitted from the surface of the CO in the APW, we substitute  $b = L/\omega_{\infty}$  to find the corresponding interior turning points  $r_-$  in terms of  $L$ . The simplest example is a radially directed ray. Using  $h = 1$  and setting  $L = 0$  eliminates the  $b$  terms, giving

$$r_{\pm} = \frac{k}{2\omega_{\infty}^2} \left( 1 \pm \sqrt{1 - \frac{8M\omega_{\infty}^2}{k}} \right). \quad (34)$$

However, as seen from the effective potential and equation 24, radially outgoing rays experience absorption at the turning points due to the plasma frequency. Thus, rays are absorbed at  $r_{\pm}$ , but all other rays with non-vanishing  $L$  are free to propagate.

We demonstrate the behaviour of trapped and deflected rays in the APW in Fig. 1. In this and the remainder of the examples in this work we assume  $h = 1$  and use  $R/r_g = 1.6$  with  $R = 3.2M$ . We assume a ray frequency near the ceiling of the APW,  $\omega_{\infty} = 0.9999\omega_{\infty 0}$ . The top panel of this figure illustrates trajectories that are emitted from the CO surface with  $L$  varying from 0 to 0.45 in increments of 0.05. The fiducial radially directed rays with  $L = 0$  reach  $r_-$ , marked as a dotted circle and are absorbed. The exact position of the absorption point of the fiducial ray is marked as an open circle. Rays with finite  $L$  reach a maximum altitude and turn back to the CO. The exterior rays which are turned away from the CO are shown with impact parameters  $b = L/\omega_{\infty}$ , using the corresponding angular momenta for the interior rays. These trajectories turn away from the CO. The closest approach for the fiducial ray  $r_+$  is marked as a dash-dotted circle, and the absorption point is explicitly marked for the external fiducial ray. The other rays are turned away from the CO and return to infinity.

The effective potential for the fiducial rays is shown in the lower panel of Fig. 1 as the thick curved line. The fiducial rays are represented by the horizontal thin lines, with the absorption points  $r_{\pm}$  marked as open circles, and the corresponding radii marked with vertical dotted and dash-dotted lines. The interior of the CO is shown in both panels by the light grey area. Thus, when observed at frequencies within the APW ( $\omega_{\infty} \leq \omega_{\infty 0}$ ), the CO is effectively ‘cloaked’ from detection by distant observers.

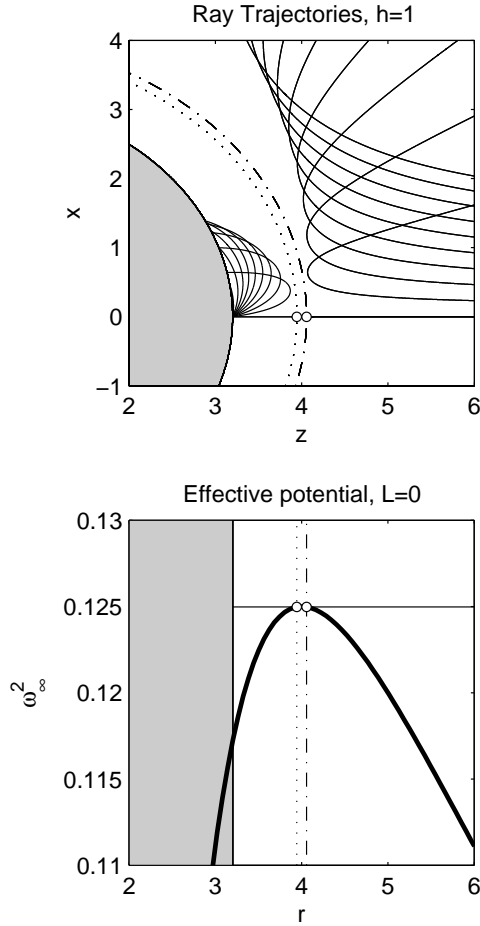
## 2.2 Maximum impact parameter for rays within the EW

The maximum impact parameter for rays at and above the high-frequency end of the EW is found by setting  $r = R$  in equation 23,

$$b_{\text{max}} = \frac{Rn(R)}{A(R)^{1/2}} \quad (35)$$

with emission angle  $\delta_{\text{max}} = \pi/2$  for a ray that makes a grazing escape from the CO surface. However, for COs sheathed in power-law distributions of plasma with  $h < 2$ , frequencies in the EW have the maximum impact parameter reduced since  $r_c > R$ . At  $\omega_0$ , the maximum impact parameter for





**Figure 1.** Examples of ray tracing in the APW. Top: rays emitted from the surface of a CO (grey disc) and passing nearby with frequency  $\omega_\infty = 0.9999\omega_{\infty 0}$ . We assume a plasma distribution with  $h = 1$ . The absorption points for radial rays are marked with open circles, and the inner turning radius at  $r_-$  is marked by a dotted circular arc. A variety of other rays are shown with angular momentum that increases in increments of 0.05, to a maximum of  $L = 0.45$ . These rays are emitted from the CO surface and reach a maximum distance  $r \leq r_-$ , before being reflected back to the surface of the CO. The exterior rays that pass by the CO have impact parameters defined as  $b = L/\omega_\infty$ , and  $r \geq r_+$ . These rays are reflected away from the CO and return to infinity. The outer absorption radius  $r_+$  (equation 34) is reached by the fiducial ray with  $b = 0$  and is marked as the dash-dotted circular arc. The absorption points for the fiducial rays are marked as open circles. Bottom: the effective potential (asymptotic plasma frequency) for the fiducial rays ( $L = 0$ ) is shown as the thick black line, and the inner and outer absorption radii marked as vertical dotted and dash-dotted lines, respectively. The points where absorption occurs are open circles and the CO interior is shaded grey. This example makes use of a highly relativistic star with  $R/r_g = 1.60$ ,  $R = 3.2M$  and a plasma frequency of the form  $\omega_e^2 \propto 1/r$  ( $h = 1$ ).

rays to escape vanishes entirely, and the image of the stellar surface to a distant observer appears to be a single point. We call the maximum impact parameter in the EW  $b_\omega$ .

In general, the maximum impact parameter is the limiting case for which equation 33 has a single solution. A ray with maximum impact parameter ( $\delta = \pi/2$ ) will enter the circular orbit at  $r_c$ . Thus, the maximum impact parameter

is found by evaluating equation 23 at  $r = r_c$ ,

$$b_\omega = \frac{r_c n(r_c)}{A(r_c)^{1/2}}. \quad (36)$$

However, this expression presents a difficulty since solving for  $r_c$  from the potential derivative requires the angular momentum  $L$ , which itself is limited by  $b_\omega$  for escaping rays at a particular frequency. Therefore, we propose to use the turning point relationship (equation 33) directly to discover the maximum impact parameter given only  $\omega_\infty < \omega_{\infty+}$  and without a priori knowledge of  $r_c$  explicitly.

For general  $h < 2$ , the solution of the turning point equation must be found using numerical methods since terms with fractional exponents will be present. Analytical solutions exist for integer values of  $h$ , however the homogeneous  $h = 0$  case simply reproduces the behaviour of a massive particle with mass  $m = \sqrt{k}$  (Kulsrud & Loeb 1992), though the general procedure used below can also provide the analytical solution. For the remainder of this section we examine the  $h = 1$  case, for which equation 33 reduces to a cubic with the general form

$$r^3 + cr^2 + dr + f = 0. \quad (37)$$

A unique solution for this equation can be found by enforcing the discriminant to vanish. This discriminant condition results in a second cubic equation in  $b^2$ , solved by standard methods (Holmes 2002). To state the solution explicitly, we define

$$C = \frac{k}{\omega_\infty^2} \left( 3M + \frac{k}{4\omega_\infty^2} \right) - 27M^2 \quad (38)$$

$$D = \frac{Mk^2}{\omega_\infty^4} \left( \frac{k}{\omega_\infty^2} - 6M \right) \quad (39)$$

$$F = \frac{M^2 k^2}{\omega_\infty^6} \left( \frac{1}{\omega_\infty^2} - 8Mk \right) \quad (40)$$

along with

$$p = D - \frac{C^2}{3} \quad (41)$$

and

$$q = \frac{2C^3 - 9CD + 27F}{27}. \quad (42)$$

In terms of these quantities, the maximum impact parameter that causes the discriminant to vanish is

$$b_\omega = \sqrt{t_0 - \frac{C}{3}}. \quad (43)$$

When  $p \neq 0$  the function  $t_0$  is given in terms of hyperbolic functions (Holmes 2002). For  $4p^3 + 27q^2 > 0$  and  $p < 0$ ,

$$t_0 = -2 \frac{q}{|q|} \sqrt{-\frac{p}{3}} \cosh \left[ \frac{1}{3} \cosh^{-1} \left( \frac{-3|q|}{2p} \sqrt{-\frac{3}{p}} \right) \right] \quad (44)$$

and for  $p > 0$ ,

$$t_0 = -2 \sqrt{\frac{p}{3}} \sinh \left[ \frac{1}{3} \sinh^{-1} \left( \frac{3q}{2p} \sqrt{\frac{3}{p}} \right) \right]. \quad (45)$$

With the maximum impact parameter we find the corresponding angular momentum for a given frequency,  $L =$

$\omega_\infty b_\omega$ , and the critical radius  $r_c$  follows. Numerical evaluation confirms that equations 43 and 36 produce identical results.

The angle that a ray with the maximum impact parameter makes with respect to the normal direction at the stellar surface is

$$\delta_\omega = \sin^{-1} \left( \frac{b_\omega}{b_{\max}} \right). \quad (46)$$

This shows that near  $\omega_0$ , rays that escape have small  $\delta_\omega$  and are close to the radial direction  $\hat{r}$ . As the observed frequency  $\omega_\infty$  increases, rays arriving at the observer will leave the CO at increasing angles, with the maximum  $\delta_{\max} = \pi/2$  at the top of the EW. We show rays with impact parameter  $b = 0.99b_\omega$  for 25 frequencies ranging logarithmically across the EW using the radial limit of  $\omega_0$  (equation 30) to  $\omega_+$  for  $h = 1$  in Fig. 2.

We demonstrate how elements of the CO surface appear on the sky of a distant observer in Fig. 3. In the top panel of this figure, we launch rays from the surface of the CO to an observer located a great distance away in the  $+z$  direction. For illustrative purposes we choose four frequencies in the EW, which we parametrize as  $\omega = \omega_0 + \eta\Delta_\omega$ , where  $\Delta_\omega = \omega_+ - \omega_0$  is the width of the EW, and  $\eta$  ranges from 0 to 1. For convenience, we will refer to these frequencies simply in terms of  $\eta$ . We choose  $\eta$  to have the arbitrary values  $10^{-3}$ ,  $10^{-2}$ ,  $10^{-1}$  and 1, respectively. For each of these frequencies, we find the maximum impact parameter  $b_\omega$  and consider the arbitrary impact parameter  $b = 0.99b_\omega$  to avoid the circular orbit at  $r_c$ . These rays are illustrated in the top panel of Fig. 3 and increase in  $\eta$  from the fiducial ray in the  $+y$  direction. Let us then define the angle between the position of the ray on the stellar surface and the fiducial direction as the surface angle. To illustrate the effect that the plasma has on the appearance of a CO to a distant observer, we shade each of the surface angle elements in the top panel of Fig. 3 with an alternating light dark scheme. Due to spherical symmetry these surface regions define shaded rings on the surface of the CO concentric with respect to the line of sight. The surface angle regions contain all impact parameters  $b \leq 0.99b_\omega$ . We plot these regions as a series of concentric discs on the lower panels. For each frequency, the surface angle regions are projected to the observer's sky. These plots show how the plasma distorts the view of the CO as a function of frequency.

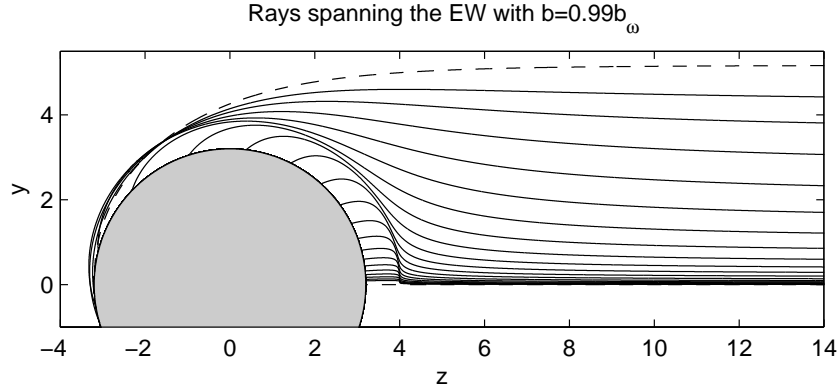
For  $\eta = 10^{-3}$ , the surface of the CO appears pointlike, and at  $\eta = 10^{-2}$  the large surface angle regions occupy a narrow range of large impact parameters and appear extremely compressed to the observer, subtending a small region at the limb of the CO disc. These areas subtend a larger solid angle on the sky as the frequency is increased. The  $\eta = 0.1$  and  $\eta = 1$  frequencies most clearly illustrate this point. Despite the maximum impact parameter for the  $\eta = 0.1$  and  $\eta = 1$  rays subtending nearly the same angle over the surface of the CO (i.e. in the top panel of Fig. 3), these regions project to different solid angles on the sky as seen in the bottom row of the figure. In these panels, the exterior dark grey ring represents the same region of the CO surface but it is markedly smaller in the  $\eta = 0.1$  panel with respect to the other solid angle elements. This shows that increasing the frequency does not simply scale the view of the CO surface to a larger size, but changes the proportions of the visible surface angle

elements relative to one another. The demagnifying nature of the relativistic images produced by COs is well known and has been extensively studied in the lensing literature (Virbhadra & Ellis 2000; Virbhadra 2009), as well as in studies of the solid angles of NS surface elements projected to an observer in vacuum conditions (Pechenick, Ftaclas & Cohen 1983; Dąbrowski & Osarczuk 1995).

We show the evolution of the pulse profile as a function of frequency in Fig. 4. The pulse profile is calculated using the method from Rogers (2015), with a single isotropically emitting radio-loud cap on the CO surface that has angular radius  $5^\circ$  centred on the pole. The pole is set to an angle  $90^\circ$  with respect to the rotation axis. This configuration aligns the line of sight and the centre of the polar cap at  $\Omega t = 0^\circ$ , where  $\Omega$  is the rotation frequency of the CO. The cap is on the opposite side of the star at  $\Omega t = 180^\circ$ . All other properties are as in Fig. 3. The curves show the evolution of the pulse profile with frequency. The large peak that occurs when the polar cap is facing away from the viewer in the vacuum case is due to multiple imaging of the cap as rays are bent around the CO. In this configuration, the polar cap appears to the observer as a bright ring around the periphery of the NS. As the visible surface area decreases with frequency, the magnitude of this peak is correspondingly reduced. At the bottom of the EW,  $\omega_0$ , the pulse profile vanishes entirely. In addition to the frequency-dependent pulse morphology, the propagation of beamed pulsar radiation through plasma also results in a frequency-dependent time delay, which is strongly affected by the plasma properties (Rogers 2015).

### 3 DISCUSSION

In the analytical examples worked in the previous sections, we have assumed a CO with  $R = 3.2M$ , slightly larger than the vacuum photon sphere at  $r_p = 3M$ . Since we find the location of a turning point with  $r_c > R$ , the maximum possible compactness of a CO is relevant. To address the question of how compact a CO can be, we examined parametrized analytical fits to 22 separate equations of state (EoS) for neutron matter (Gungor & Eksi 2012) that include a large cross-section of unique physics ranging from conventional  $npe\mu$  matter to hybrid stars that possess exotic baryon condensates in their cores. For each of these EoS fits, we solve the Tolman–Oppenheimer–Volkov equations (Glendenning 2000) to derive the mass–radius relationship. We also additionally include two examples for strange stars, based on perturbative QCD calculations (Fraga, Pisarski & Schaffner-Bielich 2001; Orsaria et al. 2011). The WFF1 equation of state (Wiringa, Fiks & Fabrocini 1988; Gungor & Eksi 2012) gives the smallest compactness ratio at  $R/r_g = 1.509$  above our lower limit of 1.5 found from equation 29. This implies that highly compact NSs that are sheathed in power-law distributions of plasma can allow for the frequency effects studied in Section 2 provided that the plasma density drops off slowly enough ( $h < 2$ ). The soft EoS that produces COs with  $R/r_g > 2$  requires  $h < 1$  to show joint effects from gravitational lensing and plasma. While our results focus specifically on the power-law case, the frequency windows exist for any plasma distribution with density that drops off sufficiently slowly.



**Figure 2.** Examples of ray tracing in the EW ( $\omega_0 < \omega \leq \omega_+$ ). Rays are emitted from the surface of a CO (grey disc). At a given frequency  $\omega$ , only rays with impact parameter less than the maximum  $b < b_\omega$  escape the CO surface. At exactly  $b = b_\omega$ , rays enter the circular orbit at  $r = r_c$ . Therefore, we arbitrarily choose rays with impact parameter slightly below the maximum,  $b = 0.99b_\omega$ . The limiting cases are the rays marked by dashed lines at  $b = 0$  and the vacuum maximum,  $b_{\text{max}}$ . Low-frequency rays have a more apparent kink in their trajectories than the high-frequency rays at  $r_c$ . This example makes use of a highly relativistic star with  $R/r_g = 1.60$ ,  $R = 3.2M$  and a plasma frequency of the form  $\omega_e^2 \propto 1/r$  ( $h = 1$ ).

The  $h = 1$  case has particular relevance for the study of pulsar wind nebulae. An aligned rotating NS produces a particle wind with a density that is proportional to  $1/r$  (Contopoulos, Kazanas & Fendt 1999; Harding, Contopoulos & Kazanas 1999; Kirk, Lyubarsky & Petri 2009; Tong, Xu, Song & Qiao 2013). Provided radio emission from low altitudes in the pulsar magnetosphere (Hassall et al. 2012; Jones 2016a), both general relativistic effects and the refractive index of the magnetospheric plasma may be relevant. These effects may also be significant for models of eclipsing binary NSs or NS-black hole systems that are nearly edge-on with respect to the observer (Keane et al. 2015; Shao et al. 2015).

Finally, let us consider the ratios of the upper and lower EW limits with respect to the plasma frequency. For the cases with  $h > 2$ , the asymptotic value of the plasma frequency is calculated with respect to the stellar surface. Evaluating equation 24 at the stellar surface gives

$$\omega_{\infty e} = \left[ A(R) \frac{k}{R^h} \right]^{1/2}. \quad (47)$$

However, when  $h < 2$ , the circular orbit radius is external to the stellar surface,  $R < r_c$ . The propagation condition requires that observable rays have frequency in excess of the asymptotic plasma frequency  $\omega_{\infty 0}$  in order to escape the CO. Thus, for  $h < 2$  the asymptotic plasma frequency must be evaluated with respect to the circular orbit radius  $r_c$  at the potential maximum. Since the most significant observable effects due to the plasma occur within the EW, it is of interest to compare the enhancement of the EW limits with respect to the asymptotic plasma frequency evaluated at the surface of the CO. Using equations 26 and 24 we find

$$\frac{\omega_{\infty+}}{\omega_{\infty e}} = \left[ \left( 1 - \frac{h}{2} \right) \frac{R - 2M}{R - 3M} \right]^{1/2}. \quad (48)$$

For a radially directed ray, we use equation 30 to find

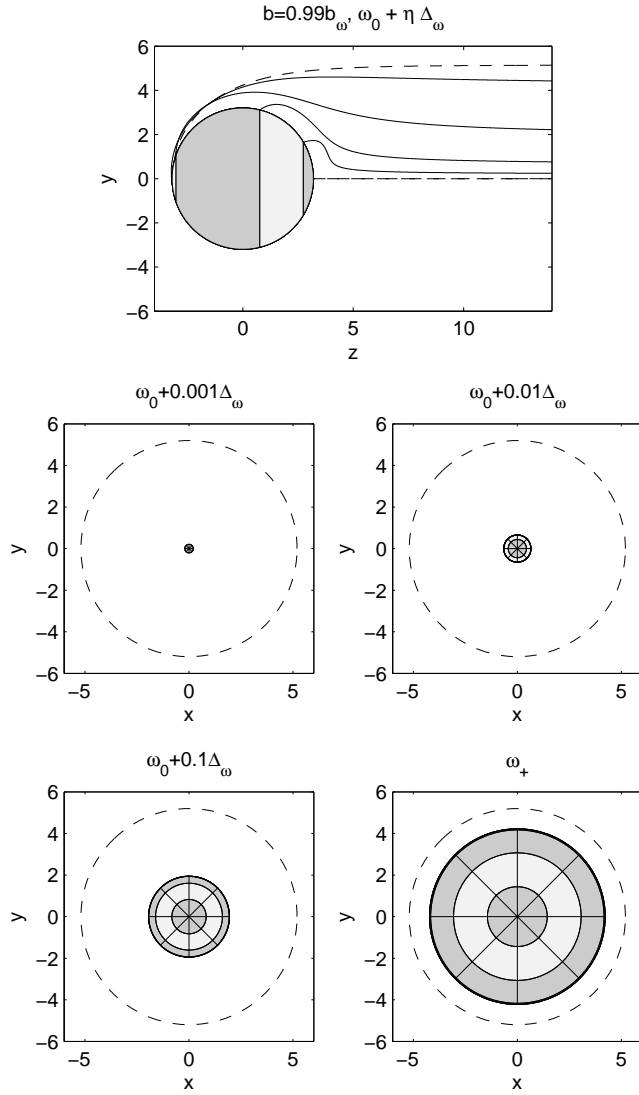
$$\frac{\omega_{\infty 0}}{\omega_{\infty e}} = \left[ \frac{1}{h+1} \frac{1}{A(R)} \left( \frac{h}{h+1} \frac{R}{2M} \right)^h \right]^{1/2}. \quad (49)$$

Note that besides the mass and radius of the CO, the ratios

depend only on the power-law exponent, such that the specific details of the plasma contained in the constant  $k$  drop out entirely.

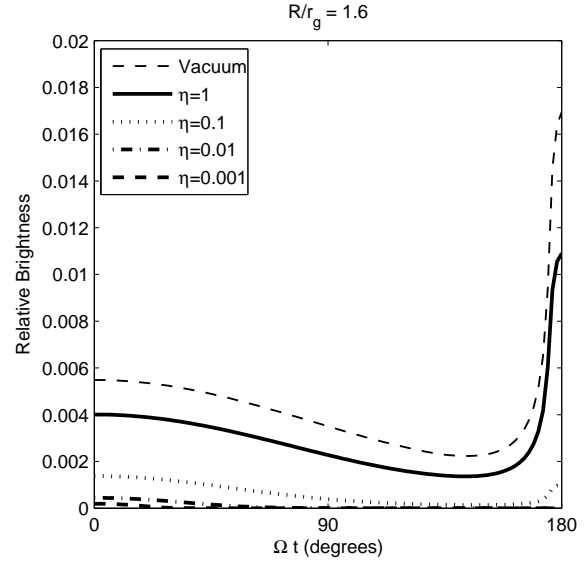
As an example, let us return to the case with  $h = 1$  and a compactness ratio  $R/r_g = 1.60$ . With these parameters, we use equation 49 to find the lower limit of the EW at  $\omega_{\infty 0} = 1.033\omega_{\infty e}$  and equation 48 evaluates to  $\omega_{\infty+} = 1.732\omega_{\infty e}$ . In Rogers (2015), we estimated an upper limit of the plasma frequency of the order of  $\sim 100$  MHz, which is boosted to  $\sim 170$  MHz from the effects presented in this work. Within this frequency range the pulse profiles should be drastically affected as shown in Fig. 4. While the increase in the cut-off frequency for  $h = 1$  is modest, in the limit  $h \rightarrow 0$  the limiting frequency ratio over the EW varies between  $\omega_{\infty 0} = 1.633\omega_{\infty e}$  and  $\omega_{\infty+} = 2.450\omega_{\infty e}$ , a moderate boost compared to the plasma frequency cutoff at the stellar surface. Therefore, due to the increase in the frequency range of the EW from a potential maximum external to the CO surface, the combined action of plasma and gravitational effects may occur in COs at slightly higher frequencies than previously estimated in the literature (Bisnovatyi-Kogan & Tsupko 2015; Rogers 2015). Despite this modification, the values above represent a rough estimate since relativistic plasma in a pulsar magnetosphere must have a plasma frequency  $\omega_e < 100$  MHz (Gedalin, Melrose & Gruman 1998; Melrose & Gedalin 1999; Jones 2016b).

The calculations in Section 2 and the pulse profiles shown in Fig. 4 are a first approximation. In general, we expect other significant factors must also be taken into account to describe real COs. The assumption that pulsed radio emission occurs near the surface is most reasonable for millisecond pulsars with small magnetospheres (Gil & Kijak 1993; Kijak & Gil 1998) and rapid rotation that causes distortions of the pulse profile through the Doppler effect (Cadeau et al. 2007). We have also treated the plasma as cold and ignored the magnetic field; however, both of these factors should play a significant role in describing the distribution of plasma within the magnetosphere and the propagation of electromagnetic radiation through it (Kulsrud & Loeb 1992; Gedalin & Melrose



**Figure 3.** Projection of CO surface elements to a distant observer. Top panel: rays within the EW with frequency  $\omega = \omega_0 + \eta \Delta\omega$ , where  $\Delta\omega = \omega_+ - \omega_0$  is the frequency range of the EW, with  $\omega_0$  taken for the fiducial ray. We set  $\eta$  to  $10^{-3}$ ,  $10^{-2}$ ,  $10^{-1}$  and 1. We then find the maximum impact parameter  $b = 0.99b_\omega$  for each of these frequencies, and plot the path from the surface to the observer (in the  $+z$  direction). The value of  $\eta$  increases from the fiducial ray in the direction of increasing  $y$ . These impact parameters and the fiducial ray define the surface angle regions, which are represented as bands over the CO surface. Lower panels: each surface angle element is projected towards the observer and show the effect that observing frequency has on the appearance of the CO by changing the projections of the apparent solid angles. This example makes use of a highly relativistic star with  $R/r_g = 1.60$ ,  $R = 3.2M$  and a plasma frequency of the form  $\omega_e^2 \propto 1/r$  ( $h = 1$ ).

2001; Petrova 2002). A complete description requires the full treatment of covariant radiative transfer, derived in Broderick & Blandford (2003a) and Broderick & Blandford (2004). In fact, polarized emission can be used as a probe of the plasma density around X-ray binaries and pulsars (Broderick & Blandford 2003a).



**Figure 4.** Pulse profiles corresponding to the example shown in Fig. 3. We use a polar cap of angular size  $5^\circ$  inclined at  $90^\circ$  with respect to the rotation axis. The centre of the polar cap aligns with the line of sight at  $\Omega t = 0^\circ$  and is on the opposite side of the CO at  $\Omega t = 180^\circ$ .

The narrow, low-frequency bands required to observe pulse profiles provide the greatest challenge to detecting the effects described in this work directly. However, there are a number of instruments suited to studying the low-frequency pulsed emission and time delays required, in particular the Low-Frequency Array (LOFAR; van Haarlem et al. 2013), which observes in the frequency range 10–240 MHz (Stappers et al. 2011), the Long Wavelength Array between 10 and 88 MHz (LWA; Stoval et al. 2015), the Ukrainian T-shaped Radio Telescope in the range of 10–30 MHz (UTR-2; Zakharenko et al. 2013) and the Murchison Widefield Array between 80 and 300 MHz (MWA; Tremblay et al. 2015). Extremely low frequency observations ( $< 30$  MHz) are complicated by a number of effects, including scattering from the interstellar medium, radio frequency interference from man-made sources and transmission through the Earth’s atmosphere. Below 10 MHz radio signals are completely attenuated by ionospheric propagation effects. Despite these difficulties, anomalous X-ray pulsars have been studied at 102–111 MHz (Malov & Malofeev 2010; Malov 2014), and a large sample of pulsar light curves have been studied over a broad range of frequencies by LOFAR (Bilous et al. 2016; Kondratiev et al. 2016), down to as low as 15 MHz (Pilia et al. 2016). Generally, a low-frequency turnover in the spectrum of radio pulsars is associated with a corresponding evolution of the pulse profile, related to processes occurring near the rest-frame plasma frequency above the polar cap (Sieber 1970; Malofeev & Malov 1980). While many NSs do not show evidence of such a turnover (Bilous et al. 2016; Kondratiev et al. 2016), this feature has been identified in a number of objects. For our purposes, we focus on the pulsars with a low-frequency turnover, such as PSR B1133+16 (Jones 2013), the millisecond pulsars J2145–0750 (Kuzmin & Losovsky 1996; Stappers 2008; Dowell et al. 2013; Kuniyoshi et al. 2015) and J1012+5307 (Kuzmin & Losovsky 2001; Kondratiev et al. 2016), as well



as a handful of additional systems (see for example Kuniyoshi et al. 2015; Kondratiev et al. 2016). We do not suggest that lensing and plasma effects account for these observations in their entirety, but if components of the radio beam arise from emission near the surface of the NS, these components can be affected by joint effects of gravity and the surrounding plasma environment.

The Square Kilometre Array (SKA; Karastergiou et al. 2015) is projected to be in operation by 2020. The sensitivity of the SKA will be vital for studying the emission properties of PSRs and detecting new systems. However, extreme measures are required to overcome the fundamental limitations of low-frequency observations imposed by artificial and ionospheric sources of radio interference. A proposed solution has been to place an antenna array composed of a swarm of satellites in a lunar orbit (Bergman et al. 2009; Boonstra et al. 2010; Baan et al. 2012; Rajan et al. 2016). In principle, such a space-based radio telescope could be sensitive to ultralow frequencies in the range 300 kHz–30 MHz, opening a new avenue of inquiry into the emission processes active in NSs.

#### 4 CONCLUSIONS

In the vacuum case, all frequencies of electromagnetic waves are affected equally by the lensing effects of the Schwarzschild metric. However, for a power-law distribution of plasma in the curved space–time around a CO, the issue of whether a star is observable at a particular frequency is more subtle, since the apparent radius of the star is frequency dependent with respect to a distant observer.

To quantify the behaviour of rays as a function of frequency, we defined two frequency windows for power-law plasma distributions with  $h < 2$ : the EW is the window in which rays originating from the CO surface can escape to a distant observer,  $\omega_0 < \omega \leq \omega_+$ , and the APW spans the frequency range  $\omega_- < \omega \leq \omega_0$ . In the APW, a family of rays exist that are emitted from and return to the CO surface. These behaviours are dependent on the presence of a plasma that drops off more slowly than the  $r^{-2}$  contribution from the vacuum term in the effective potential. The frequency windows are found analytically for power-law distributions and have analytical expressions for integer  $h < 2$ , but must be solved numerically for non-integer values. The maximum impact parameter for rays in the EW is given as a function of frequency from analysis of the classical turning points of the effective potential.

Finally, we studied plasma effects on the appearance of a CO formed on the sky of a distant observer. For a range of angles on the surface of a CO, we found the corresponding projection on the observer’s sky. The solid angle subtended by the CO appears to change apparent size when the observing frequency is increased. At frequencies much larger than the upper limit of the EW, vacuum behaviour is recovered.

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